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# On Stabilization of Nonlinear Distributed Parameter Port-Controlled Hamiltonian Systems via Energy-Shaping<sup>1</sup>

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## Abstract

Energy-shaping techniques have been successfully used for stabilization of nonlinear finite dimensional systems for 20 years now. In particular, for systems described by Port-Controlled Hamiltonian (PCH) models, the “control by interconnection” method provides a simple and elegant procedure for stabilization of nonlinear systems with finite dissipation. In this paper we explore the possibility of extending this technique to the case where the plant contains a distributed parameter subsystem, in the form of a transmission line between the plant and the controller. **Note:** The present paper is an abridged version of [3].

## 1 Introduction

Passivity-based control (PBC) of finite dimensional systems was introduced already 20 years ago, and has reached a good level of maturity with many different variations and successful applications, see e.g. [4, 1, 2] for a list of references. The basic underlying principles of PBC is to shape the total energy of the system, which is clearly independent of the dimension of the state space. Hence, it seems natural to look for possible extensions of PBC to the distributed setting; this is the topic that we address in the present paper.

The PBC design technique that we consider here is the “control by interconnection” method developed for regulation of Port-Controlled Hamiltonian (PCH) systems<sup>1</sup> in [2], see also Section 4.3.1 of [4]. As thoroughly detailed in the aforementioned references, and

briefly explained in Section 3 below, the central component of this approach is the generation of Casimir functions, which are dynamical invariants independent of the Hamiltonian function that allow us to achieve the energy shaping objective.

Instrumental for our developments is the notion of a Dirac structure, that formalizes in a geometric language the concept of power conserving interconnection for PCH systems. Dirac structures for finite dimensional *implicit* PCH systems are reported in [4]. Recently, the framework was extended to distributed parameter systems in [6]. The main contribution of this paper is the derivation of the conditions for existence of the Casimir functions for a controller-infinite dimensional subsystem-plant configuration.

In the next section we present the Dirac structure associated to PCH models, for reference we start with finite dimensional systems, and then present the infinite dimensional case. Section 4 contains our main result, first, we present the interconnection between lumped and distributed parameter systems described above. Then, we find the conditions that guarantee the existence of Casimir functions for the interconnected system. Finally, we outline an stabilization procedure for the interconnected PCH system based in the results of [5, 9] and we present a controller design example, and then wrap up the paper with some concluding remarks in Section 5.

## 2 Dirac Structures and Port Controlled Hamiltonian Systems

It has been shown in [4, 6] that the notion of power preserving interconnection can be formalized geometrically by a Dirac structure, which is a subspace of the space of efforts and flows. In this section we briefly present this concept for lumped systems as well as

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<sup>2</sup>PCH systems are a natural extension of classical Hamiltonian systems to consider the existence of external variables, see [4].

for distributed parameter systems with a single scalar spatial variable ranging in a segment  $[0, l]$ .

## 2.1 Lumped Parameter Systems

To define the notion of a Dirac structure for lumped parameter systems we consider the finite dimensional linear space  $\mathcal{F}$  of flows  $f$ , and its dual,  $\mathcal{F}^*$ , which is the space of efforts  $e$ . Power is then defined as  $P = \langle e | f \rangle$  with  $\langle \cdot | \cdot \rangle$  denoting the duality product.<sup>2</sup> As shown in [4], on  $\mathcal{F} \times \mathcal{F}^*$  there exists a canonically defined symmetric bilinear form

$$\ll (f_1, e_1), (f_2, e_2) \gg \triangleq \langle e_1 | f_2 \rangle + \langle e_2 | f_1 \rangle \quad (2.1)$$

**Definition 1** A (constant) Dirac structure on the finite dimensional linear space  $\mathcal{F}$  is a linear subspace  $S \subset \mathcal{F} \times \mathcal{F}^*$  such that  $S^\perp = S$ , where  $S^\perp$  denotes the orthogonal complement of  $S$  with respect to the bilinear form (2.1).

As an immediate corollary of the definition we see that for all  $(f, e) \in \mathcal{D}$  we have that  $\langle e, f \rangle = 0$ . Hence, a Dirac structure defines a power conserving relation. Here, we are interested in PCH systems where the Dirac structure can be represented in input-output form as

$$\begin{aligned} \mathcal{D} &\triangleq \{(f, e) \in \mathcal{F} \times \mathcal{F}^* \mid e_P = g^\top(x) e_S, \\ e_R &= g_R^\top(x) e_S, f_S = -\mathcal{J}(x) e_S - g_R(x) f_R - g(x) f_P\} \end{aligned} \quad (2.2)$$

where  $(f, e) := (f_S, f_R, f_P, e_S, e_R, e_P)$ <sup>3</sup>,  $x \in \mathbb{R}^n$  is the systems state,  $\mathcal{J}(x) = -\mathcal{J}^\top(x)$  is the so-called interconnection matrix, and  $g(x)$ ,  $g_S(x)$ ,  $g_R(x)$  are input matrices of suitable dimensions. From (2.2) it is easy to show that  $\mathcal{D} = \mathcal{D}^\perp$ . Furthermore, given that for all  $(f, e) \in \mathcal{D} = \mathcal{D}^\perp$ , we have  $0 = \ll (f, e), (f, e) \gg = 2\langle f | e \rangle$ , therefore  $P = 0$  for all elements of  $\mathcal{D}$ , consequently, the Dirac structure defines a power conserving relation between the effort and flow variables. If we assume that the flow and effort variables of the dissipative elements are related by  $f_R = -R(x) e_R$ , where  $R(x) = R^\top(x) \geq 0$ , we obtain the following relationship between the power variables of the PCH system

$$\begin{bmatrix} f_S \\ e_P \end{bmatrix} = \begin{bmatrix} -\mathcal{J}(x) + \mathcal{R}(x) & -g(x) \\ g^\top(x) & 0 \end{bmatrix} \begin{bmatrix} e_S \\ f_P \end{bmatrix} \quad (2.3)$$

where  $\mathcal{R}(x) := g_R(x)^\top R(x) g_R(x)$ . We call this the power variables representation of a PCH system. To

<sup>2</sup>If  $\mathcal{F}$  is a Hilbert space, then  $\mathcal{F}^*$  can be naturally identified with  $\mathcal{F}$  in such a way that for all  $f \in \mathcal{F}$ ,  $e \in \mathcal{F}^*$  we have  $\langle e | f \rangle = \langle e, f \rangle$ , where  $\langle \cdot, \cdot \rangle$  is the standard inner product; see e.g. [8].

<sup>3</sup> $f_S$ ,  $f_R$ ,  $f_P$  are the stored, the dissipated and the external flows respectively.

recover the classical—energy variables—state-space description of the PCH system we recall that in these variables the energy stored by the conservative elements is defined by the Hamiltonian function,  $H(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ , in such a way that the increase of energy equals power, that is,  $P(t) = \frac{d}{dt} H[x(t)] = \langle \frac{\partial H}{\partial x}[x(t)] | \dot{x}(t) \rangle$ , hence, the flow and effort variables of the energy-storing elements are given by<sup>4</sup>  $f_S = -\dot{x}$  and  $e_S = \frac{\partial H}{\partial x}(x)$ . In this way, we get the well-known model of a PCH system

$$\begin{aligned} \dot{x} &= [\mathcal{J}(x) - \mathcal{R}(x)] \frac{\partial H}{\partial x}(x) + g(x) f_P \\ e_P &= g^\top(x) \frac{\partial H}{\partial x}(x) \end{aligned} \quad (2.4)$$

which clearly satisfies the energy balance

$$\dot{H}[x(t)] = -\frac{\partial^\top H}{\partial x}[x(t)] \mathcal{R}[x(t)] \frac{\partial H}{\partial x}[x(t)] + e_P^\top(t) f_P(t)$$

in the following, we will use  $e_P = y_p$  and  $f_P = u_p$  in order to relate to the standard input-output notation.

## 2.2 Distributed Parameter Systems

In order to define the Dirac structure for such a distributed parameter system we have to consider, instead of a *finite-dimensional* linear space  $\mathcal{F} \times \mathcal{F}^*$  as in the lumped parameter case, an infinite-dimensional function space; see [6] for a more general differential-geometric setting appropriate to multi-dimensional spatial domains.

This function space  $\mathcal{F} \times \mathcal{F}^*$  will be defined as follows. Consider the function space  $\mathcal{E} \triangleq \mathcal{H}_{1M}(Z) \times \mathcal{H}_{1E}(Z) \times B$ , with  $\mathcal{H}_{1M}(Z)$ ,  $\mathcal{H}_{1E}(Z)$  denoting the space of magnetic and electric efforts,  $e_M$  and  $e_E$ , respectively, and  $B$  denoting the external efforts  $e_b$  at the boundary of  $Z$ . Here  $\mathcal{H}_1(Z)$  denotes the Sobolev space of  $\mathcal{L}_2$  functions on  $Z$  whose derivatives are also in  $\mathcal{L}_2$ . Then  $\mathcal{F}$  is defined as the dual space of  $\mathcal{E}$  with respect to the duality product (defining again the power  $P$ )

$$\langle (e_E, e_M, e_b), (f_E, f_M, f_b) \rangle = \int_0^\ell [f_E(z) e_E(z) + f_M(z) e_M(z)] dz + e_b f_b \Big|_0^\ell$$

with  $(f_E, f_M, f_b)$  denoting respectively the electric flow, the magnetic flow (both functions on  $Z$  belonging to the dual Sobolev space  $\mathcal{H}_1(Z)^*$ ), and the boundary flow. In analogy with (2.1), we can define a bilinear form between two elements of  $\mathcal{F} \times \mathcal{E}$  as

$$\ll (f^1, e^1), (f^2, e^2) \gg \triangleq \int_0^\ell (e_E^1 f_E^2 + e_E^2 f_E^1 + e_M^1 f_M^2 + e_M^2 f_M^1) dz + (e_b^1 f_b^2 + e_b^2 f_b^1) \Big|_0^\ell \quad (2.5)$$

<sup>4</sup>Strictly speaking, the power variables  $f_S, e_S$  do not live in a constant linear space but instead in the tangent and co-tangent spaces to the finite dimensional manifold of energy variables. This is formalized with the definition of a non-constant Dirac structure on a manifold, see [7] for details.

where we have defined  $(f^i, e^i) \triangleq (f_E^i, f_M^i, f_b^i, e_E^i, e_M^i, e_b^i) \in \mathcal{F} \times \mathcal{E}$ ,  $i = 1, 2$ .

The proposition below, whose proof may be found in [3], defines the Dirac structure for the case of infinite dimensional systems with scalar spatial variables.

**Proposition 1** Define the following subspace of  $\mathcal{F} \times \mathcal{E}$

$$\mathcal{D} \triangleq \{(f, e) \in \mathcal{F} \times \mathcal{E} \mid \begin{bmatrix} f_E \\ f_M \\ f_b \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e_E \\ e_M \\ e_{Eb} \\ e_{Mb} \end{bmatrix}\} \quad (2.6)$$

Then  $\mathcal{D}$  is a constant Dirac structure, with respect to the bilinear form (2.5).

We will now prove that, equivalently to the finite dimensional case, the elements of the Dirac structure of Proposition 1 satisfy a generalized form of power conservation. Indeed, from (2.5) we have that  $\ll (f, e), (f, e) \gg = 0$ , for all  $(f \mid e) \in \mathcal{D}$ , that is,  $2 \int_0^\ell [e_E f_E + e_M f_M] dz + 2e_b f_b \big|_0^\ell = 0$ , thus, we get the energy balance property  $\int_0^\ell [e_E f_E + e_M f_M] dz = e_b(0)f_b(0) - e_b(\ell)f_b(\ell)$ , which says that the total power in the domain  $Z$  is equal to the power ingoing at the boundary 0 minus the power outgoing at the boundary  $\ell$ .

The distributed parameter PCH system in power variables follows directly from the Dirac structure. Evaluating the lower relation of (2.6) at the boundary points  $\{0, \ell\}$ , we get

$$f_0 = -e_{M0}, \quad e_0 = e_{E0}, \quad f_\ell = -e_{M\ell}, \quad e_\ell = e_{E\ell} \quad (2.7)$$

In order to write (2.6) in energy variables, we consider the Hamiltonian density  $H : \mathcal{H}_{1M}^* \times \mathcal{H}_{1E}^* \times Z \rightarrow \mathcal{L}_1$ , associated with the total energy functional  $\mathcal{H} = \int_0^\ell H(\bar{q})dz$ , with  $\bar{q} \triangleq [q_E, q_M, z]$ . We assume  $\mathcal{H}$  to be differentiable, with time derivative [5]

$$\frac{d\mathcal{H}}{dt} = \int_0^\ell [\delta_M^\top \mathcal{H} \quad \delta_E^\top \mathcal{H}]^\top \begin{bmatrix} \frac{\partial q_M}{\partial t} \\ \frac{\partial q_E}{\partial t} \end{bmatrix} dz,$$

where we have introduced  $\delta_E \mathcal{H} = \frac{\delta \mathcal{H}}{\delta q_E}$ ,  $\delta_M \mathcal{H} = \frac{\delta \mathcal{H}}{\delta q_M}$  to denote the variational derivative. As in the lumped parameter case, the power and energy variables are related by  $f_E = -\frac{\partial}{\partial t} q_E$ ,  $f_M = -\frac{\partial}{\partial t} q_M$ ,  $e_E = \delta_E \mathcal{H}$  and  $e_M = \delta_M \mathcal{H}$  and the distributed parameter PCH system (2.6) can be written in energy variables as

$$\begin{bmatrix} \frac{\partial}{\partial t} q_E \\ \frac{\partial}{\partial t} q_M \\ f_b \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \delta_E \mathcal{H} \\ \delta_M \mathcal{H} \\ e_{Eb} \\ e_{Mb} \end{bmatrix}, \quad (2.8)$$

which satisfies the energy balancing equation  $\frac{d\mathcal{H}}{dt} = \delta_E^\top \mathcal{H}(0) \delta_M \mathcal{H}(0) - \delta_E^\top \mathcal{H}(\ell) \delta_M \mathcal{H}(\ell)$ .

### 2.3 Example: Transmission Line

In this subsection we present the PCH model of a transmission line whose dynamics are described by the well-known telegrapher's equations. In the Dirac framework the model is given as follows : The energy variables are electric charge and magnetic flux,  $q_E(t) = q(z, t)$ ,  $q_M(t) = \lambda(z, t)$ , respectively. The total energy functional becomes  $\mathcal{H} = \frac{1}{2} \int_0^\ell \left[ \frac{q^2(z, t)}{C_{ii}(z)} + \frac{\lambda^2(z, t)}{L_{ii}(z)} \right] dz$ . Then, the telegrapher's equations may be expressed as a distributed PCH system of the form (2.8), that is

$$\begin{bmatrix} \frac{\partial}{\partial t} q(z, t) \\ \frac{\partial}{\partial t} \lambda(z, t) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} \frac{q(z, t)}{C_{ii}(z)} \\ \frac{\lambda(z, t)}{L_{ii}(z)} \end{bmatrix}, \quad (2.9)$$

$$f_b = -\frac{\lambda(z, t)}{L_{ii}(z)} \bigg|_0^\ell, \quad e_b = \frac{q(z, t)}{C_{ii}(z)} \bigg|_0^\ell$$

the lower relation of (2.9) defines voltages and currents at the boundary points  $\{0, \ell\}$ . In the following we assume that the physical parameters of the transmission line are upper and lower bounded in  $[0, \ell]$ , that is,  $L_m \leq \frac{1}{L_{ii}(z)} \leq L_M$ ,  $C_m \leq \frac{1}{C_{ii}(z)} \leq C_M$  with  $L_i, C_i > 0$ ,  $i = M, m$ .

### 3 Control by Interconnection : Finite Dimensional Case

In this section we first briefly review the PBC design method of "control by interconnection" for lumped parameter systems, and then present its extension to the infinite dimensional case. A key step in this method, that allow us to achieve the energy shaping objective, is the generation of Casimir functions.

In the "control by interconnection" method we consider a PCH plant described by (2.4) in interconnection with a PCH controller with state  $x_c \in \mathbb{R}^m$ , input  $u_c$ , output  $y_c$ , and  $H_c(x_c)$  the energy of the controller.<sup>5</sup> In power variables the PCH controller is described by

$$\begin{bmatrix} f_c \\ y_c \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e_c \\ u_c \end{bmatrix} \quad (3.1)$$

The interconnection constraints are power-preserving of the form

$$u_c = y_p, \quad u_p = -y_c \quad (3.2)$$

The composed system is clearly still Hamiltonian and can be written in power variables as

$$\begin{bmatrix} f_s \\ f_c \end{bmatrix} = \begin{bmatrix} -\mathcal{J}(x) + \mathcal{R}(x) & -g(x) \\ g^\top(x) & 0 \end{bmatrix} \begin{bmatrix} e_s \\ e_c \end{bmatrix} \quad (3.3)$$

<sup>5</sup>See [4] for an explanation for the choice of this structure of the PCH controller.

with  $H_{cl}(\chi) \triangleq H(x) + H_c(x_c)$  the *closed-loop* energy function (defined in an extended state space  $\chi \triangleq [x, x_c]^\top$ ). We can easily see that this energy function is non-increasing, since  $\frac{d}{dt}H_{cl} = -\frac{\partial^\top H}{\partial x}(x)\mathcal{R}(x)\frac{\partial H}{\partial x}(x) \leq 0$ , and we would like to shape it to assign a minimum at the desired point. However, although  $H_c(x_c)$  can be freely assigned, the systems energy-function  $H(x)$  is given, and its not clear how can we effectively shape the overall energy. One possibility is to restrict the motion of the closed-loop system to a certain subspace of  $\chi$ , say  $\Omega \subset \mathbb{R}^{n+m}$ , by rendering  $\Omega$  *invariant*.<sup>6</sup> In this way, we can express the closed-loop total energy as a function of  $x$  only. In the Energy-Casimir method [2], we look for dynamical invariants which are *independent* of the Hamiltonian function. More precisely we look for functions  $\mathcal{C}(\chi)$  –called Casimir functions– such that along the dynamics of the PCH system  $\frac{d}{dt}\mathcal{C}(\chi) = 0$  *independent* of the energy function. Without loss of generality, we consider Casimir functions of the form  $\mathcal{C}(\chi) = F(x) - x_c$ . Since we want these functions to remain constant along the trajectories of the closed-loop dynamics (3.3) *irrespective* of the precise form of  $H_{cl}(\chi)$ , they should be solutions of the PDEs

$$\begin{bmatrix} \frac{\partial^\top F}{\partial x}(x) \\ -I_m \end{bmatrix}^\top \begin{bmatrix} J(x) - \mathcal{R}(x) & -g(x) \\ g^\top(x) & 0 \end{bmatrix} = 0 \quad (3.4)$$

It is clear that the level sets  $\Omega \triangleq \{\chi | x_c = F(x) + \kappa\}$ , with  $\kappa$  a constant that can be set to zero without loss of generality, are invariant sets for the closed-loop system, hence the closed-loop total energy defined now in the restricted state space  $\chi_r \triangleq \chi|_\Omega$  becomes  $H_d(\chi_r) \triangleq H(x) + H_c[F(x)]$ . This function can now be shaped with a suitable selection of the controller energy  $H_c(x_c)$ .

In [2] necessary and sufficient conditions for the existence of the Casimir functions, i.e., for solvability of the PDEs (3.4), are given. Before closing this subsection we make the following important observation that will be instrumental to extend the notion of Casimir function to the distributed case. From the power variable description of the PCH system (3.3) we see that the Casimir functions are determined by the subspace  $\{e \in \mathcal{F}^* | \langle 0, e \rangle \in \mathcal{D}\} \subset \mathcal{F}^*$ . Indeed,  $\mathcal{C}(x, x_c)$  is a Casimir function if and only if

$$(0, \frac{\partial \mathcal{C}}{\partial \chi}(\chi)) \in \mathcal{D} \quad (3.5)$$

<sup>6</sup>A set  $\Omega \subset \mathbb{R}^{n+m}$  is invariant if the following implication holds:  $\chi(0) \in \Omega \Rightarrow \chi(t) \in \Omega, \forall t \geq 0$ .

### 3.1 Example: Control by Interconnection of RLC Circuit

To illustrate the control by interconnection method let us consider a RLC circuit described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -R \end{bmatrix} \begin{bmatrix} \frac{x_1}{C} \\ \frac{x_2}{L} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_p \quad (3.6)$$

$y_p = \frac{x_2}{L}$

where  $x_1$  is the charge in the capacitor and  $x_2$  is the flux in the inductor, in power preserving interconnection (3.2) with the *PCH controller* (3.1). The control objective is to stabilize (3.6) at the equilibrium point  $x_* = [\frac{x_{1*}}{C}, 0]^\top$ . It is easy to verify that a function which satisfies (3.4) is given by  $\mathcal{C}(\chi) = x_1 - x_c$ , therefore, at the invariant set  $\Omega \triangleq \{\chi | x_c = x_1\}$ , the closed-loop energy is given by  $H_d(\chi_r) = H(x) + H_c(x_1)$ .

The next step of the “control by interconnection” methodology is to shape the closed-loop energy in the restricted state space  $\chi_r = [x_1, x_2]^\top$ , in such a way that it has a minimum at  $x_*$ , therefore, we require that  $\frac{\partial H_d}{\partial x}(x_*) = 0$ ,  $\frac{\partial^2 H_d}{\partial x^2}(x_*) \geq 0$ . It can be shown that selecting  $H_c(x_c) = \frac{1}{2C_c} \tilde{x}_c^2 - \frac{x_{1*}}{C_c} \tilde{x}_c$ ,  $C_c > 0$  and  $\tilde{x}_c = x_c - x_{c*}$ .  $H_d(\chi_r)$  has a minimum at  $x_*$ . Finally the PCH controller is given by

$$\begin{aligned} \dot{x}_c &= \frac{x_2}{L} \\ y_c &= \frac{1}{C_c} x_c - \frac{C + C_c}{C_c C} x_{1*} \end{aligned} \quad (3.7)$$

## 4 Control by Interconnection : Mixed Finite and Infinite Dimensional Case

In this case we consider a PCH plant in interconnection with a PCH controller through a infinite dimensional system described . In order to make clear the interconnection we will work with their PCH models in power variables (2.3), (3.1) and (2.6) respectively. The interconnection constraints are of the form

$$y_c = f_0, \quad u_c = e_0, \quad y_p = e_\ell, \quad u_p = -f_\ell \quad (4.1)$$

This interconnection constraints are power preserving in the sense that if  $\ell = 0$  they become the power preserving interconnection (3.2) between the plant and the controller . In order to get the closed-loop dynamics we replace the interconnection constraints (4.1) into (2.3), (3.1) and (2.6), and we obtain

$$\begin{aligned} \begin{bmatrix} f_s \\ f_c \end{bmatrix} &= \begin{bmatrix} -\mathcal{J}(x) + \mathcal{R}(x) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_s \\ e_c \end{bmatrix} - \begin{bmatrix} g(x)e_{M\ell} \\ e_{E0} \end{bmatrix} \\ \begin{bmatrix} f_E \\ f_M \end{bmatrix} &= \begin{bmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} e_E \\ e_M \end{bmatrix} \\ e_{E\ell} &= g(x)^T e_s, \quad e_{M0} = -e_c \end{aligned} \quad (4.2)$$

where we have used (2.7). The closed-loop energy defined in the extended space  $\chi = [x, x_c, q_E(z, t), q_M(z, t)]^\top$  is given by  $H_{cl}(\chi) = H(x) + H_c(x_c) + \mathcal{H}(\bar{q})$ , with energy rate equals to  $\dot{H}_{cl}(\chi) = -\frac{\partial^\top H}{\partial x}(x)\mathcal{R}(x)\frac{\partial H}{\partial x}(x)$ .

#### 4.1 Casimir Functions

Now, we look for the Casimir functions of the system dynamics, to this end, we will use the Casimir function definition (3.5). Hence a function  $\mathcal{C}(\chi)$  will be a Casimir function provided

$$\begin{aligned} & \left[ \mathcal{J}(x) + \mathcal{R}(x) \right] \frac{\partial}{\partial x} \mathcal{C}(\chi) + g(x) \delta_M \mathcal{C}(\chi) \big|_{t=0} = 0 \\ & -\delta_E \mathcal{C}(\chi) \big|_0 = 0 \\ & \delta_M \mathcal{C}(\chi) = 0, \quad \delta_E \mathcal{C}(\chi) = 0 \\ & \delta_E \mathcal{C}(\chi) \big|_t = g^\top(x) \frac{\partial}{\partial x} \mathcal{C}(\chi), \quad \delta_M \mathcal{C}(\chi) \big|_0 = -\frac{\partial}{\partial x_c} \mathcal{C}(\chi) \end{aligned} \quad (4.3)$$

from the third and fourth relations of (4.3), we can conclude that, every Casimir function of (4.2) should be linear with respect to the spatial variables, that is

$$\begin{aligned} \delta_M \mathcal{C}(\chi) &= \text{constant as a function of } z \\ \delta_E \mathcal{C}(\chi) &= \text{constant as a function of } z \end{aligned} \quad (4.4)$$

so that,

$$\begin{aligned} \delta_M \mathcal{C}(\chi) &= \delta_M \mathcal{C}(\chi) \big|_0 = \delta_M \mathcal{C}(\chi) \big|_t = -\frac{\partial}{\partial x_c} \mathcal{C}(\chi) \\ \delta_E \mathcal{C}(\chi) &= \delta_E \mathcal{C}(\chi) \big|_0 = \delta_E \mathcal{C}(\chi) \big|_t = g(x)^\top \frac{\partial}{\partial x} \mathcal{C}(\chi) \end{aligned} \quad (4.5)$$

replacing (4.5) into (4.3) and considering (4.4), condition (4.3) reduces to

$$\begin{bmatrix} \mathcal{J}(x) + \mathcal{R}(x) & -g(x) \\ g^\top(x) & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \mathcal{C}(\chi) \\ \frac{\partial}{\partial x_c} \mathcal{C}(\chi) \end{bmatrix} = 0$$

$$\delta_E \mathcal{C}(\chi) = g^\top(x) \frac{\partial}{\partial x} \mathcal{C}(\chi), \quad \delta_M \mathcal{C}(\chi) = -\frac{\partial}{\partial x_c} \mathcal{C}(\chi)$$

From the ‘‘control by interconnection’’ point of view, we are interested in Casimir functions relating the state variables of the interconnected system, in such a way that we can define an invariant set  $\Omega$  like in subsection 3. In particular we consider Casimir functions of the form

$$\mathcal{C}(\chi) = -x_c + F(x) + \mathcal{F}(\bar{q}(z, t)) \quad (4.6)$$

that means, we are looking for functions which satisfy the following conditions

$$\mathcal{R}(x) \frac{\partial F}{\partial x}(x) = 0, \quad \frac{\partial^\top F}{\partial x}(x) \mathcal{J}(x) = g^\top(x). \quad (4.7)$$

and

$$\delta_E \mathcal{F}(\bar{q}(z, t)) = 0, \quad \delta_M \mathcal{F}(\bar{q}(z, t)) = 1 \quad (4.8)$$

Hence we have proved

**Proposition 2** *The functions  $-x_c + F(x) + \mathcal{F}(\bar{q}(z, t))$  are Casimir functions of the interconnected PCH system (4.2) if and only if the function  $F(x)$  satisfies (4.7) and the functional  $\mathcal{F}(\bar{q}(z, t))$  satisfies (4.4) and (4.8) if  $y_p = e_t$  or*

$$\delta_E \mathcal{F}(\bar{q}(z, t)) = 1, \quad \delta_M \mathcal{F}(\bar{q}(z, t)) = 0 \quad (4.9)$$

if  $y_p = f_t$ .

**Proof:** See [3] for the proof of (4.9). ■

Hence, we can define  $\chi_r$  as the state space restricted to the invariant set  $\Omega = \{\chi \mid x_c = F(x) + \mathcal{F}(\bar{q}(z, t))\}$ .

#### 4.2 Control Design

The idea behind the stability argument of distributed parameter systems is the same of lumped parameter systems in that we wish to show that the equilibrium solution corresponds to a strict extremum of the total energy, with the difference that in distributed parameter systems care must be taken to specify the norm associated with the stability argument because stability with respect to one norm does not necessarily imply stability with respect to another norm.<sup>7</sup> In the case of mixed lumped and distributed parameter systems we will define stability in the sense of Lyapunov as follows

**Definition 2** *The equilibrium point  $\chi_{r*}$  of a mixed lumped and distributed parameters system is said to be stable in the sense of Lyapunov with respect to the norm  $\|\cdot\|$ , if for every  $\epsilon > 0$  there exist  $\delta > 0$  such that  $\|\chi_r(0) - \chi_{r*}\| < \delta \Rightarrow \|\chi_r - \chi_{r*}\| < \epsilon$  for all  $t > 0$ , where  $\chi_r(0)$  is the initial condition of  $\chi_r$ .*

The mathematical procedure to show stability can be summarized as follows (see for details [5]) : Take as candidate Lyapunov function the closed-loop energy restricted to  $\Omega$ , show that it has an extremum at  $\chi_{r*}$  and give conditions to assure that the extremum is a minimum. Finally, asymptotic stability can be shown using the infinite dimensional version of La Salle theorem [9].

#### 4.3 Example: RLC with a Transmission Line

To illustrate the control by interconnection of mixed finite and infinite dimensional systems we consider the example of Section 3 but now we insert between the controller and the RLC circuit a transmission line as

<sup>7</sup>This is a consequence of the fact that in an infinite dimensional space not every convergent sequence on the unit ball converges to a point on the unit ball, that is unit balls in infinite dimensional spaces need not be compact [5].

we can see in Fig.1. The control objective is to stabilize the RCL circuit to the equilibrium point  $x_*$  defined in subsection (3.1). In this case the power pre-

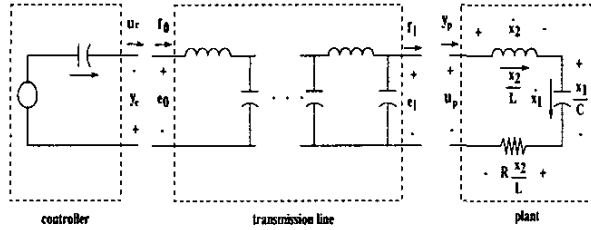


Figure 1: Interconnection constraints.

serving interconnection constraints are given by

$$y_c = -e_0, u_c = f_0, y_p = f_l, u_p = e_l \quad (4.10)$$

The RCL circuit described by (3.6), the controller described by (3.1) and the transmission line modeled by (2.6), under the interconnection constraints (4.10), give the following interconnected dynamics in power variables

$$\begin{bmatrix} \dot{f}_s \\ \dot{f}_c \end{bmatrix} = \begin{bmatrix} -\mathcal{J}(x) + \mathcal{R}(x) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_s \\ e_c \end{bmatrix} - \begin{bmatrix} g(x)e_{E\ell} \\ -e_{M0} \end{bmatrix} \\ \begin{bmatrix} \dot{f}_E \\ \dot{f}_M \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} e_E \\ e_M \end{bmatrix} \\ e_{M\ell} = -g(x)^T e_s, \quad e_{E0} = -e_c \quad (4.11)$$

From Proposition 2, we have that a Casimir function of (4.11) is given by  $\mathcal{C}(\chi) = -x_c + x_1 + \int_0^\ell q(z, t) dz$ . Then, in the invariant set  $\Omega = \{\chi \mid x_c = x_1 + \int_0^\ell q(z, t) dz\}$  the closed-loop energy is equal to  $H_d(\chi_r) = \frac{1}{2} \frac{x_1^2}{C} + \frac{1}{2} \frac{x_2^2}{L} + H_c(x_1 + \int_0^\ell q(z, t) dz) + \frac{1}{2} \int_0^\ell \left( \frac{q^2(z, t)}{C_{ll}(z)} + \frac{\lambda^2(z, t)}{L_{ll}(z)} \right) dz$ . Hence, following the procedure proposed in [5] we can establish the next result, whose proof is given in [3]

**Proposition 3** Consider the RCL circuit defined by (3.6), the transmission line modeled by (2.9), and the PCH controller defined by

$$\dot{x}_c = -\frac{\lambda(0, t)}{L_{ll}(0)} \\ y_c = \frac{1}{C_c} x_c - \frac{C + C_c}{C_c C} x_{1*} - \frac{x_{1*}}{C C_c} \int_0^\ell C_{ll}(z) dz \quad (4.12)$$

under the interconnection constraints (4.10). The resulting interconnected system has an stable equilibrium in the sense of Definition 2 at

$$\chi_{r*} = \left[ \frac{x_{1*}}{C}, 0, \frac{x_{1*}}{C} C_{ll}(z), 0 \right]^T, \quad \chi_r = [x, \bar{q}(z, t)]^T$$

with respect to the norm  $\|\chi_r\| = \left( \Delta x_1^2 + \Delta x_2^2 + \int_0^\ell \Delta q^2(z, t) dz + \int_0^\ell \Delta \lambda^2(z, t) dz \right)^{\frac{1}{2}}$ .

Comparing the PCH controller of the lumped parameter case (3.7) and the PCH controller defined by (4.12) we can see that the effect of the transmission line is compensated by the mixed nature of  $x_c$  in (4.12).

## 5 Conclusions

In this article, we have presented a first stage to extend the PBC method to stabilize mixed finite and infinite dimensional systems. The presented approach relies in the generation of Casimir functions for the closed-loop dynamics and the control by interconnection introduced in [2], taking into account the peculiarities due to the infinite dimensional nature of the interconnected system.

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